THE STRUCTURE OF THE SPACE OF AFFINE KÄHLER CURVATURE TENSORS AS A COMPLEX MODULE

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ABSTRACT. We use results of Matzeu and Nikčević to decompose the space of affine Kähler curvature tensors as a direct sum of irreducible modules in the complex setting.
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1. INTRODUCTION

1.1. **Curvature decompositions.** We begin by giving a brief history and overview of the theory of curvature decompositions to put the main result of this paper in the proper setting. Such decompositions are central to the theory of modern differential geometry. Consequently, the subject is a vast one and we can only sketch a few of the highlights. The decompositions in general stabilize; there is a crucial dimension m_0 so that if the dimension m exceeds m_0 then the number of summands is constant; one obtains the decomposition in lower dimensions by setting certain of the summands to $\{0\}$. Singer and Thorpe [23] showed that the space $\mathfrak{R}$ of Riemann curvature tensors has 3 irreducible components under the action of the orthogonal group $\mathcal{O}$ in dimension $m \geq 4$; these are the space of Weyl conformal curvature tensors, the space of trace free Ricci tensors, and the space of constant sectional curvature tensors. There are only 2 components in dimension 3 and only 1 component in dimension 2. Tricerri and Vanhecke [25] gave a similar decomposition of $\mathfrak{R}$ in the almost Hermitian setting; the appropriate structure group there is the unitary group $\mathcal{U}^\star$ and there are 10 irreducible unitary modules comprising the decomposition in dimension $m \geq 8$; if $m = 6$, then there are 9 summands and if $m = 4$, then there are 7 summands in the decomposition. If one assumes that the complex structure involved is in fact integrable, Gray [12] showed one of the components does not appear so there are 9 irreducible unitary modules in the decomposition in the context of Hermitian geometry if $m \geq 8$, 8 if $m = 6$, and 6 if $m = 4$. Kähler geometry remains a field of active investigation in many different contexts [16, 17, 21, 26]; the Riemannian Kähler curvature tensors have 3 factors in their decomposition ($m \geq 4$) as unitary modules. Note that Sasakian geometry is intimately linked with Kähler geometry – see, for example, the discussion in [6, 8] – so odd dimensional phenomena can also appear in this setting. De Smedt [7] showed there are 37 modules in the decomposition of $\mathfrak{R}$ under the action of the symplectic group in the hyper-Hermitian setting for $m \geq 16$ (the number drops to 36 if $m = 12$ and to 32 if $m = 8$). Hyper-Kähler geometry also is being actively studied – see, for example [5, 11, 20].

Although not a curvature decomposition, the following decomposition is in the same spirit. Let $\nabla\Omega$ be the covariant derivative of the Kähler form on an almost Hermitian manifold. Gray and Hervella [13] showed that $\nabla\Omega$ can be decomposed into 4 separate components if $m \geq 6$ and 2 components if $m = 4$; this gives rise to the celebrated $16 = 2^4$ classes of almost Hermitian manifolds. We also refer to subsequent results of Brozos-Vázquez et al. [4] in the almost pseudo-Hermitian and in the almost para-Hermitian settings.

Weyl geometry is in a certain sense midway between Riemannian and affine geometry. Higa [14, 15] decomposed the space of Weyl curvature tensors into irreducible orthogonal modules; there are 4 summands if $m\geq 4$. We refer to [1, 9, 10] for further details in this regard. Strichartz [24] decomposed the space of affine curvature tensors as a direct sum of 3 modules over the general linear group GL if $m\geq 3$; we present his result in Theorem 1.1 below. Subsequently, Bokan [2] decomposed this space as an orthogonal module; there are 8 summands if $m\geq 4$, 6 summands if $m=3$, and 3 summands if $m=2$. This decomposition is perhaps less natural since an auxiliary inner product needs to be introduced. Matzeu and Nikčević [18, 19] generalized Bokan's work to decompose the space of Kähler affine curvature tensors $\mathcal{K}$ as a unitary module; there are 12 summands in the decomposition if $m\geq 6$ and 10 summands in the decomposition if $m=4$. This result will be presented as Theorem 1.5. In this present paper, we use Theorem 1.5 to establish Theorem 1.2 which generalizes Theorem 1.1 to the complex setting; there are 6 summands in the decomposition for $m\geq 4$.

1.2. **Affine structures.** We now introduce the requisite notation to state the results of [18, 19, 24] and the main result of this paper more precisely. An *affine manifold* is a pair (M,∇) where M is a smooth manifold and where ∇ is a torsion free connection on the tangent bundle TM. We refer to [22] for further information concerning affine geometry. The associated *curvature operator* $\mathcal{R}\in\otimes^2T^*M\otimes\operatorname{End}(TM)$ is defined by setting:

$$\mathcal{R}(x,y):=\nabla_x\nabla_y-\nabla_y\nabla_x-\nabla_{[x,y]}\,.$$

This tensor satisfies the following identities:

$$\mathcal{R}(x,y)=-\mathcal{R}(y,x)\text{ and }\mathcal{R}(x,y)z+\mathcal{R}(y,z)x+\mathcal{R}(z,x)y=0\,. \tag{1.a}$$

It is convenient to work in a purely algebraic context. Let V be a real m-dimensional vector space. We say that $A\in\otimes^2V^*\otimes\operatorname{End}(V)$ is an *affine curvature operator* if A has the symmetries given above in Equation (1.a). Let $\mathfrak{A}$ be the subspace of all such operators.

The natural structure group in this setting is the *general linear group* GL. The Ricci tensor ρ is a GL equivariant map from $\mathfrak{A}$ to $V^*\otimes V^*$ defined by setting:

$$\rho(x,y):=\operatorname{Tr}\{z\to\mathcal{R}(z,x)y\}\,.$$

We decompose $\otimes^2V^*=\Lambda^2\oplus S^2$ into the space of alternating 2-tensors Λ^2 and the space of symmetric 2-tensors S^2. We summarize below the fundamental decomposition of the space of affine curvature operators $\mathfrak{A}$ under the natural action of the general linear group [24]:

Theorem 1.1. *If $m\geq 3$, then $\mathfrak{A}\approx\{\mathfrak{A}\cap\ker(\rho)\}\oplus\Lambda^2\oplus S^2$ as a* GL *module where $\{\mathfrak{A}\cap\ker(\rho),\Lambda^2,S^2\}$ are inequivalent and irreducible* GL *modules.*

1.3. **Affine Kähler Structures.** The triple (M,J,∇) is said to be an *affine Kähler manifold* if J is an almost complex structure on M (i.e. an endomorphism of the tangent bundle TM so that $J^2=-\operatorname{id}$), if ∇ is a torsion free connection on TM, and if $\nabla J=0$; necessarily the complex structure is integrable in this setting. The curvature operator $\mathcal{R}$ then satisfies the additional symmetry:

$$J\mathcal{R}(x,y)=\mathcal{R}(x,y)J\quad\text{for all}\quad x,y\,. \tag{1.b}$$

We pass to the algebraic context. Let J be a complex structure on a real vector space V. We consider the subgroup of all linear maps commuting or anti-commuting with J:

$$\mathrm{GL}_{\mathbb{C}}^{\star}=\{\Xi\in\mathrm{GL}:\Xi J=\pm J\Xi\}\,.$$

We set $\chi(\Xi)=\pm1$ to define a $\mathbb{Z}_2$ representation of $\mathrm{GL}_{\mathbb{C}}^\star$ into $\mathbb{Z}_2$. We shall allow into consideration maps which replace J by $-J$ as the two complex structures J and $-J$ play interchangable roles in many geometric settings; the group $\mathrm{GL}_{\mathbb{C}}^\star$ is a $\mathbb{Z}_2$ extension of the usual complex general group.

The space of Kähler affine tensors is defined by imposing the Kähler identity given in the geometric setting by Equation (1.b), namely:

$$\mathcal{K}:=\{\mathcal{A}\in\mathfrak{A}:\mathcal{A}(v_1,v_2)J=J\mathcal{A}(v_1,v_2)\ \forall v_1,v_2\in V\}\,.$$

J acts by pullback on tensors of all types. We may decompose $\Lambda^2=\Lambda^2_+\oplus\Lambda^2_-$, $S^2=S^2_+\oplus S^2_-$, and $\mathcal{K}=\mathcal{K}_+\oplus\mathcal{K}_-$ where

$$\begin{aligned}
\mathcal{K}_\pm&:=\{\mathcal{A}\in\mathcal{K}:\mathcal{A}(Jv_1,Jv_2)=\pm\mathcal{A}(v_1,v_2)\ \forall v_1,v_2\in V\},\\
\Lambda^2_\pm&:=\{\psi\in\Lambda^2:\psi(Jv_1,Jv_2)=\pm\psi(v_1,v_2)\ \forall v_1,v_2\in V\},\\
S^2_\pm&:=\{\phi\in S^2:\phi(Jv_1,Jv_2)=\pm\phi(v_1,v_2)\ \forall v_1,v_2\in V\}\,.
\end{aligned}$$

Since J appears an even number of times, these are $\mathrm{GL}_{\mathbb{C}}^\star$ modules and the Ricci tensor defines short exact sequences of $\mathrm{GL}_{\mathbb{C}}^\star$ modules:

$$0\rightarrow\mathcal{K}_\pm\cap\ker(\rho)\rightarrow\mathcal{K}_\pm\xrightarrow{\rho}\Lambda^2_\pm\oplus S^2_\pm\rightarrow0\,.$$

It will follow from Lemma 2.2 that this sequence is split in the category of $\mathrm{GL}_{\mathbb{C}}^\star$ modules; the following result generalizes Theorem 1.1 to this setting and is the main result of this paper:

Theorem 1.2. *If $m\geq6$, then we have the following isomorphisms decomposing $\mathcal{K}_\pm$ as the direct sum of irreducible and inequivalent $\mathrm{GL}_{\mathbb{C}}^\star$ modules:*

$$\mathcal{K}_\pm\approx\{\mathcal{K}_\pm\cap\ker(\rho)\}\oplus\Lambda^2_\pm\oplus S^2_\pm\,.$$

Remark 1.3. The modules $\{\mathcal{K}_+\cap\ker(\rho),\mathcal{K}_-\cap\ker(\rho),S^2_+,S^2_-,\Lambda^2_-\}$ have different dimensions and are therefore inequivalent. Since S^2_+ is not isomorphic to Λ^2_+ as a $\mathcal{U}^\star$ module (see Theorem 1.5 below), the modules appearing in Theorem 1.2 are inequivalent. If $m=4$, the same decomposition pertains if we set the module $\mathcal{K}_-\cap\ker(\rho)=\{0\}$ and therefore delete this module from consideration.

1.4. **The Matzeu-Nikčević decomposition.** The proof we shall give of Theorem 1.2 rests on results of [18, 19]. We assume given an auxiliary positive definite inner product $\langle\cdot,\cdot\rangle$ on V so that $J^*\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle$; the triple $(V,\langle\cdot,\cdot\rangle,J)$ is said to be a *Hermitian vector space*. The orthogonal and unitary groups are then defined by setting:

$$\mathcal{O}:=\{T\in\mathrm{GL}:T^*\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle\}\quad\text{and}\quad\mathcal{U}^\star:=\mathcal{O}\cap\mathrm{GL}_{\mathbb{C}}^\star\,.$$

We use the metric to raise and lower indices. We may now regard:

$$\begin{aligned}
\mathcal{K}&:=\{A\in\mathfrak{A}:A(x,y,z,w)=A(x,y,Jz,Jw)\},\\
\mathcal{K}_\pm&:=\{A\in\mathcal{K}:A(Jx,Jy,z,w)=\pm A(x,y,z,w)\}\,.
\end{aligned}$$

The decomposition of $\mathcal{K}$ as a unitary module is given in [18, 19]; it extends easily to give a $\mathcal{U}^\star$ module decomposition as well. We first introduce some auxiliary notation:

Definition 1.4. Let $(V,\langle\cdot,\cdot\rangle,J)$ be a Hermitian vector space. Let $\{e_i\}$ be an orthonormal basis for V. Adopt the *Einstein convention* and sum over repeated indices to define:.

(1) $\rho_{13}(A)(x,y)=A(e_i,x,e_i,y)$ and $\rho(A)(x,y)=A(e_i,x,y,e_i)$.
(2) $\Omega(x,y):=\langle x,Jy\rangle$.
(3) $S^2_{0,+}:=\{\phi\in S^2_+:\phi\perp\langle\cdot,\cdot\rangle\}$ and $\Lambda^2_{0,+}:=\{\psi\in\Lambda^2_+:\psi\perp\Omega\}$.
(4) $W_9:=\{A\in\mathcal{K}_+:A(x,y,z,w)=-A(x,y,w,z)\}\cap\ker(\rho)$.
(5) $W_{10}:=\{A\in\mathcal{K}_+:A(x,y,z,w)=A(x,y,w,z)\}\cap\ker(\rho)$.

(6) $W_{11} := \mathcal{K}_+ \cap W_9^\perp \cap W_{10}^\perp \cap \ker(\rho_{13}) \cap \ker(\rho)$.
(7) $W_{12} := \mathcal{K}_- \cap \ker(\rho)$, $\tau := A(e_i, e_j, e_j, e_i)$, and $\tau_J := \varepsilon^{il}\varepsilon^{jk} A(e_i, Je_j, e_k, e_l)$.

Theorem 1.5. *Let $m \geq 6$. We have decompositions of the following modules as the direct sum of irreducible and inequivalent $\mathcal{U}^\star$ modules:*

$$\begin{aligned}
&\mathcal{K} \approx \mathbb{R} \oplus \chi \oplus 2 \cdot S^2_{0,+} \oplus 2\Lambda^2_{0,+} \oplus \Lambda^2_- \oplus S^2_- \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12},\\
&\mathcal{K}_+ \approx \mathbb{R} \oplus \chi \oplus 2S^2_{0,+} \oplus 2\Lambda^2_{0,+} \oplus W_9 \oplus W_{10} \oplus W_{11},\\
&\mathcal{K}_+ \cap \ker(\rho) \approx S^2_{0,+} \oplus \Lambda^2_{0,+} \oplus W_9 \oplus W_{10} \oplus W_{11},\\
&\mathcal{K}_- \approx \Lambda^2_- \oplus S^2_- \oplus W_{11},\\
&\mathcal{K}_- \cap \ker(\rho) \approx W_{11}\,.
\end{aligned}$$

Remark 1.6. We note that $\mathcal{K}_- \cap \ker(\rho) = W_{12}$ is an irreducible $\mathcal{U}^\star$ module. The decomposition of Theorem 1.5 is also into irreducible $\mathcal{U}$ modules. However, $S^2_{0,+}$ is isomorphic to $\Lambda^2_{0,+}$ as a $\mathcal{U}$ module and W_9 is isomorphic to W_{10} as a $\mathcal{U}$ module. The corresponding decompositions if $m = 4$ are obtained by setting $W_{11} = W_{12} = \{0\}$.

1.5. **Outline of the paper.** In Section 2, we shall construct a $\mathrm{GL}^\star_{\mathbb{C}}$ splitting of the map defined by the Ricci tensor ρ from $\mathcal{K}$ to $\otimes^2 V^*$. We use this splitting together with Theorem 1.5 to reduce the proof of Theorem 1.2 to the assertion that $\mathcal{K}_+ \cap \ker(\rho)$ is an irreducible $\mathrm{GL}^\star_{\mathbb{C}}$ module. In Section 3, we examine ρ_{13} and construct the orthogonal projectors from $\mathcal{K}_+$ to the subspaces of $\mathcal{K}_+ \cap \ker(\rho)$ which are isomorphic to $S^2_{0,+}$ and $\Lambda^2_{0,+}$ in Theorem 1.5. In Section 4, we use the conjugate tensor to examine the orthogonal projectors on the subspaces W_9, W_{10}, and W_{11} of Theorem 1.5. In Section 5, we complete the proof of Theorem 1.5 by showing $\mathcal{K}_+ \cap \ker(\rho)$ is an irreducible $\mathrm{GL}^\star_{\mathbb{C}}$ module.

2. The geometry of ρ

The Ricci tensor defines a $\mathrm{GL}^\star_{\mathbb{C}}$ module morphism $\rho : \mathcal{K} \to \otimes^2 V^*$ that restricts to $\mathrm{GL}^\star_{\mathbb{C}}$ module morphisms from $\mathcal{K}_\pm$ to $\Lambda^2_\pm \oplus S^2_\pm$. In this section, we shall construct a $\mathrm{GL}^\star_{\mathbb{C}}$ module morphism splitting of ρ. We first introduce some additional notation:

Definition 2.1. Let J be a complex structure on V. For $\phi_1 \in S^2_+$, $\phi_2 \in S^2_-$, $\phi_3 \in \Lambda^2_+$, and $\phi_4 \in \Lambda^2_-$ define:

$$(\sigma_1\phi_1)(x,y)z := \phi_1(x,z)y - \phi_1(y,z)x - \phi_1(x,Jz)Jy + \phi_1(y,Jz)Jx - 2\phi_1(x,Jy)Jz.$$

$$(\sigma_2\phi_2)(x,y)z := \phi_2(x,z)y - \phi_2(y,z)x - \phi_2(x,Jz)Jy + \phi_2(y,Jz)Jx.$$

$$(\sigma_3\phi_3)(x,y)z := \phi_3(x,z)y - \phi_3(y,z)x + 2\phi_3(x,y)z - \phi_3(x,Jz)Jy + \phi_3(y,Jz)Jx,$$

$$(\sigma_4\phi_4)(x,y)z := \phi_4(x,z)y - \phi_4(y,z)x + 2\phi_4(x,y)z - \phi_4(x,Jz)Jy + \phi_4(y,Jz)Jx -2\phi_4(x,Jy)Jz.$$

Since J appears an even number of times, these are $\mathrm{GL}^\star_{\mathbb{C}}$ module morphisms.

Lemma 2.2.

(1) *If $\phi_1 \in S^2_+$, then $\sigma_1\phi_1 \in \mathcal{K}_+$ and $\rho\,\sigma_1\phi_1 = -(m+2)\phi_1$.*
(2) *If $\phi_2 \in S^2_-$, then $\sigma_2\phi_2 \in \mathcal{K}_-$ and $\rho\,\sigma_2\phi_2 = (2-m)\phi_2$.*
(3) *If $\phi_3 \in \Lambda^2_+$, then $\sigma_3\phi_3 \in \mathcal{K}_+$ and $\rho\,\sigma_3\phi_3 = -(m+2)\phi_3$.*
(4) *If $\phi_4 \in \Lambda^2_-$, then $\sigma_4\phi_4 \in \mathcal{K}_-$ and $\rho\,\sigma_4\phi_4 = -(2+m)\phi_4$.*

Proof. We begin with some basic parity observations:

$$
\begin{aligned}
&\phi_1(x,Jy)=\phi_1(Jx,JJy)=-\phi_1(Jx,y),\\
&\phi_2(x,Jy)=-\phi_2(Jx,JJy)=\phi_2(Jx,y),\\
&\phi_3(x,Jy)=\phi_3(Jx,JJy)=-\phi_3(Jx,y),\\
&\phi_4(x,Jy)=-\phi_4(Jx,JJy)=\phi_4(Jx,y).
\end{aligned}
$$

It now follows that the tensors $\{\sigma_1\phi_1,\sigma_2\phi_2,\sigma_3\phi_3,\sigma_4\phi_4\}$ are anti-symmetric in the first two arguments. We verify that the Bianchi identity is satisfied by these tensors and therefore that they belong to $\mathfrak{A}$ by computing:

$$
\begin{aligned}
&(\sigma_1\phi_1)(x,y)z+(\sigma_1\phi_1)(y,z)x+(\sigma_1\phi_1)(z,x)y\\
&\quad=\phi_1(x,z)y-\phi_1(y,z)x-\phi_1(x,Jz)Jy+\phi_1(y,Jz)Jx-2\phi_1(x,Jy)Jz\\
&\quad+\phi_1(y,x)z-\phi_1(z,x)y-\phi_1(y,Jx)Jz+\phi_1(z,Jx)Jy-2\phi_1(y,Jz)Jx\\
&\quad+\phi_1(z,y)x-\phi_1(x,y)z-\phi_1(z,Jy)Jx+\phi_1(x,Jy)Jz-2\phi_1(z,Jx)Jy=0,\\
&(\sigma_2\phi_2)(x,y)z+(\sigma_2\phi_2)(y,z)x+(\sigma_2\phi_2)(z,x)y\\
&\quad=\phi_2(x,z)y-\phi_2(y,z)x-\phi_2(x,Jz)Jy+\phi_2(y,Jz)Jx\\
&\quad+\phi_2(y,x)z-\phi_2(z,x)y-\phi_2(y,Jx)Jz+\phi_2(z,Jx)Jy\\
&\quad+\phi_2(z,y)x-\phi_2(x,y)z-\phi_2(z,Jy)Jx+\phi_2(x,Jy)Jz=0,\\
&(\sigma_3\phi_3)(x,y)z+(\sigma_3\phi_3)(y,z)x+(\sigma_3\phi_3)(z,x)y\\
&\quad=\phi_3(x,z)y-\phi_3(y,z)x+2\phi_3(x,y)z-\phi_3(x,Jz)Jy+\phi_3(y,Jz)Jx\\
&\quad+\phi_3(y,x)z-\phi_3(z,x)y+2\phi_3(y,z)x-\phi_3(y,Jx)Jz+\phi_3(z,Jx)Jy\\
&\quad+\phi_3(z,y)x-\phi_3(x,y)z+2\phi_3(z,x)y-\phi_3(z,Jy)Jx+\phi_3(x,Jy)Jz=0,\\
&(\sigma_4\phi_4)(x,y)z+(\sigma_4\phi_4)(y,z)x+(\sigma_4\phi_4)(z,x)y\\
&\quad=\phi_4(x,z)y-\phi_4(y,z)x+2\phi_4(x,y)z\\
&\quad+\phi_4(y,x)z-\phi_4(z,x)y+2\phi_4(y,z)x\\
&\quad+\phi_4(z,y)x-\phi_4(x,y)z+2\phi_4(z,x)y\\
&\quad-\phi_4(x,Jz)Jy+\phi_4(y,Jz)Jx-2\phi_4(x,Jy)Jz\\
&\quad-\phi_4(y,Jx)Jz+\phi_4(z,Jx)Jy-2\phi_4(y,Jz)Jx\\
&\quad-\phi_4(z,Jy)Jx+\phi_4(x,Jy)Jz-2\phi_4(z,Jx)Jy=0.
\end{aligned}
$$

We verify these endomorphisms commute with J and belong to $\mathcal{K}$ by comparing:

$$
\begin{aligned}
&(\sigma_1\phi_1)(x,y)Jz=\phi_1(x,Jz)y-\phi_1(y,Jz)x-\phi_1(x,JJz)Jy\\
&\quad+\phi_1(y,JJz)Jx-2\phi_1(x,Jy)JJz,\\
&(J\sigma_1\phi_1)(x,y)z=\phi_1(x,z)Jy-\phi_1(y,z)Jx-\phi_1(x,Jz)JJy\\
&\quad+\phi_1(y,Jz)JJx-2\phi_1(x,Jy)JJz,\\
&(\sigma_2\phi_2)(x,y)Jz=\phi_2(x,Jz)y-\phi_2(y,Jz)x-\phi_2(x,JJz)Jy+\phi_2(y,JJz)Jx,\\
&(J\sigma_2\phi_2)(x,y)z=\phi_2(x,z)Jy-\phi_2(y,z)Jx-\phi_2(x,Jz)JJy+\phi_2(y,Jz)JJx,\\
&(\sigma_3\phi_3)(x,y)Jz=\phi_3(x,Jz)y-\phi_3(y,Jz)x\\
&\quad+2\phi_3(x,y)Jz-\phi_3(x,JJz)Jy+\phi_3(y,JJz)Jx,\\
&(J\sigma_3\phi_3)(x,y)z=\phi_3(x,z)Jy-\phi_3(y,z)Jx\\
&\quad+2\phi_3(x,y)Jz-\phi_3(x,Jz)JJy+\phi_3(y,Jz)JJx,\\
&(\sigma_4\phi_4)(x,y)Jz=\phi_4(x,Jz)y-\phi_4(y,Jz)x\\
&\quad+2\phi_4(x,y)Jz-\phi_4(x,JJz)Jy+\phi_4(y,JJz)Jx-2\phi_4(x,Jy)JJz,\\
&(J\sigma_4\phi_4)(x,y)z=\phi_4(x,z)Jy-\phi_4(y,z)Jx\\
&\quad+2\phi_4(x,y)Jz-\phi_4(x,Jz)JJy+\phi_4(y,Jz)JJx-2\phi_4(x,Jy)JJz.
\end{aligned}
$$

Let $\{e_i\}$ be a basis for V and let $\{e^i\}$ be the corresponding dual basis for V^*. We have $e^i(Je_i)=\mathrm{Tr}(J)=0$. We examine the Ricci tensor:

$$\begin{aligned}
&(\rho\sigma_1\phi_1)(y,z)=\phi_1(e_i,z)e^i(y)-\phi_1(y,z)e^i(e_i)-\phi_1(e_i,Jz)e^i(Jy)\\
&\quad+\phi_1(y,Jz)e^i(Je_i)-2\phi_1(e_i,Jy)e^i(Jz)\\
&\quad=\phi_1(y,z)-m\phi_1(y,z)-\phi_1(Jy,Jz)+0-2\phi_1(Jz,Jy)=-(m+2)\phi_1(y,z),\\
&(\rho\sigma_2\phi_2)(y,z)=\phi_2(e_i,z)e^i(y)-\phi_2(y,z)e^i(e_i)\\
&\quad-\phi_2(e_i,Jz)e^i(Jy)+\phi_2(y,Jz)e^i(Je_i)\\
&\quad=\phi_2(y,z)-m\phi_2(y,z)-\phi_2(Jy,Jz)+0=(2-m)\phi_2(y,z),\\
&(\rho\sigma_3\phi_3)(y,z)=\phi_3(e_i,z)e^i(y)-\phi_3(y,z)e^i(e_i)\\
&\quad+2\phi_3(e_i,y)e^i(z)-\phi_3(e_i,Jz)e^i(Jy)+\phi_3(y,Jz)e^i(Je_i)\\
&\quad=\phi_3(y,z)-m\phi_3(y,z)+2\phi_3(z,y)-\phi_3(Jy,Jz)+0=-(m+2)\phi_3(y,z),\\
&(\rho\sigma_4\phi_4)(y,z)=\phi_4(e_i,z)e^i(y)-\phi_4(y,z)e^i(e_i)\\
&\quad+2\phi_4(e_i,y)e^i(z)-\phi_4(e_i,Jz)e^i(Jy)+\phi_4(y,Jz)e^i(Je_i)-2\phi_4(e_i,Jy)e^i(Jz)\\
&\quad=\phi_4(y,z)-m\phi_4(y,z)+2\phi_4(z,y)-\phi_4(Jy,Jz)\\
&\quad+0-2\phi_4(Jz,Jy)=(-2-m)\phi_4(y,z).
\end{aligned}$$

The fact that $\sigma_i\phi_i$ takes values in the appropriate subspaces $\mathcal{K}_\star$ now follows from Theorem 1.5; it can also, of course, be checked directly. □

Remark 2.3. Let $m\geq 6$. We use Lemma 2.2 to split ρ and see that there is a $\mathrm{GL}_{\mathbb{C}}^\star$ module decomposition of

$$\mathcal{K}_\pm\approx\Lambda^2_\pm\oplus S^2_\pm\oplus\{\mathcal{K}_\pm\cap\ker(\rho)\}\,.$$

By Theorem 1.5, $\{\Lambda^2_+,\Lambda^2_-,S^2_+,S^2_-,\mathcal{K}_-,\mathcal{K}_+\}$ are inequivalent and non-trivial $\mathcal{U}^\star$ modules and hence, necessarily, inequivalent $\mathrm{GL}_{\mathbb{C}}^\star$ modules as well. Theorem 1.5 also yields that $\{\Lambda^2_+,\Lambda^2_-,S^2_+,S^2_-,\mathcal{K}_-\}$ are irreducible as $\mathcal{U}^\star$ modules and hence are irreducible as $\mathrm{GL}_{\mathbb{C}}^\star$ modules as well. Thus to complete the proof of Theorem 1.2, it suffices to show that $\mathcal{K}_+\cap\ker(\rho)$ is an irreducible $\mathrm{GL}_{\mathbb{C}}^\star$ module; this will be done in Lemma 5.1 after first establishing some preliminary algebraic results in Section 3 and in Section 4. The case $m=4$ is handled by setting $\mathcal{K}_-\cap\ker(\rho)=W_{12}=\{0\}$ and deleting this module from the discussion below.

3. The geometry of ρ_{13}

If $\phi\in V^*\otimes V^*$, then set:

$$\begin{aligned}
&\vartheta(\phi)(x,y,z,w):=\phi(x,w)\langle y,z\rangle-\phi(y,w)\langle x,z\rangle\\
&\quad+\phi(x,Jw)\langle y,Jz\rangle-\phi(y,Jw)\langle x,Jz\rangle-2\phi(z,Jw)\langle x,Jy\rangle.
\end{aligned}$$

Lemma 3.1. *Let $\phi\in S^2_{0,+}\oplus\Lambda^2_{0,+}$, let $\phi_1\in S^2_{0,+}$, and let $\phi_3\in\Lambda^2_{0,+}$.*

1. *$\vartheta\phi\in\mathcal{K}_+$.*
2. *$\rho\sigma_1\phi_1=-(m+2)\phi_1$ and $\rho_{13}\sigma_1\phi_1=2\phi_1$.*
3. *$\rho\sigma_3\phi_3=-(m+2)\phi_3$ and $\rho_{13}\sigma_3\phi_3=-2\phi_3$.*
4. *$\rho\vartheta\phi_1=2\phi_1$ and $\rho_{13}\vartheta\phi_1=-(m+2)\phi_1$.*
5. *$\rho\vartheta\phi_3=-2\phi_3$ and $\rho_{13}\vartheta\phi_3=-(m+2)\phi_3$.*

Proof. It is immediate from the definition that $\vartheta(\phi)$ is anti-symmetric in the first 2 arguments. Note that

$$\phi(x,Jy)=\phi(Jx,JJy)=-\phi(Jx,y)\,.$$

We verify that $\vartheta\phi$ satisfies the Bianchi identity by computing:

$$\begin{aligned}
&\vartheta(\phi)(x,y,z,w)+\vartheta(\phi)(y,z,x,w)+\vartheta(\phi)(z,x,y,w)\\
&\quad=\phi(x,w)\langle y,z\rangle-\phi(y,w)\langle x,z\rangle\\
&\quad+\phi(y,w)\langle z,x\rangle-\phi(z,w)\langle y,x\rangle\\
&\quad+\phi(z,w)\langle x,y\rangle-\phi(x,w)\langle z,y\rangle
\end{aligned}$$

$$\begin{aligned}&+\phi(x,Jw)\langle y,Jz\rangle-\phi(y,Jw)\langle x,Jz\rangle-2\phi(z,Jw)\langle x,Jy\rangle\\&+\phi(y,Jw)\langle z,Jx\rangle-\phi(z,Jw)\langle y,Jx\rangle-2\phi(x,Jw)\langle y,Jz\rangle\\&+\phi(z,Jw)\langle x,Jy\rangle-\phi(x,Jw)\langle z,Jy\rangle-2\phi(y,Jw)\langle z,Jx\rangle=0.\end{aligned}$$

We will show that $\vartheta\phi\in\mathcal{K}_+$ by demonstrating that:

$$\vartheta\phi(x,y,z,w)=\vartheta\phi(x,y,Jz,Jw)=\vartheta\phi(Jx,Jy,z,w).$$

We compare:

$$\begin{aligned}&\vartheta(\phi)(x,y,z,w)=\phi(x,w)\langle y,z\rangle-\phi(y,w)\langle x,z\rangle\\&\quad+\phi(x,Jw)\langle y,Jz\rangle-\phi(y,Jw)\langle x,Jz\rangle-2\phi(z,Jw)\langle x,Jy\rangle,\\&\vartheta(\phi)(x,y,Jz,Jw)=\phi(x,Jw)\langle y,Jz\rangle-\phi(y,Jw)\langle x,Jz\rangle\\&\quad+\phi(x,JJw)\langle y,JJz\rangle-\phi(y,JJw)\langle x,JJz\rangle-2\phi(Jz,JJw)\langle x,Jy\rangle,\\&\vartheta(\phi)(Jx,Jy,z,w)=\phi(Jx,w)\langle Jy,z\rangle-\phi(Jy,w)\langle Jx,z\rangle\\&\quad+\phi(Jx,Jw)\langle Jy,Jz\rangle-\phi(Jy,Jw)\langle Jx,Jz\rangle-2\phi(z,Jw)\langle Jx,JJy\rangle.\end{aligned}$$

We use Lemma 2.2 to determine $\rho\sigma_1$ and $\rho\sigma_3$. We compute $\rho\vartheta$:

$$\begin{aligned}&\rho\vartheta(\phi)(y,z)=\varepsilon^{il}\phi(e_i,e_l)\langle y,z\rangle-\varepsilon^{il}\phi(y,e_l)\langle e_i,z\rangle\\&\quad+\varepsilon^{il}\phi(e_i,Je_l)\langle y,Jz\rangle-\varepsilon^{il}\phi(y,Je_l)\langle e_i,Jz\rangle-2\varepsilon^{il}\phi(z,Je_l)\langle e_i,Jy\rangle\\&\quad=0-\phi(y,z)+0-\phi(y,JJz)-2\phi(z,JJy)\\&\quad=-\phi(y,z)+\phi(y,z)+2\phi(z,y)=2\phi(z,y).\end{aligned}$$

We examine ρ_{13}. Let $\varepsilon_{ij}=\langle e_i,e_j\rangle$. Since $\phi\perp\langle\cdot,\cdot\rangle$ and since $\phi\perp\Omega$, $\varepsilon^{il}\phi(e_i,e_l)=0$ and $\varepsilon^{il}\phi(e_i,Je_l)=0$.

$$\begin{aligned}&(\rho_{13}\vartheta\phi)(y,w)=\varepsilon^{ik}\phi(e_i,w)\langle y,e_k\rangle-\varepsilon^{ik}\phi(y,w)\langle e_i,e_k\rangle\\&\quad+\varepsilon^{ik}\phi(e_i,Jw)\langle y,Je_k\rangle-\varepsilon^{ik}\phi(y,Jw)\langle e_i,Je_k\rangle-2\varepsilon^{ik}\phi(e_k,Jw)\langle e_i,Jy\rangle\\&\quad=\phi(y,w)-m\phi(y,w)-\phi(Jy,Jw)-0-2\phi(Jy,Jw)=-(m+2)\phi(y,w),\\&(\rho_{13}\sigma_1\phi_1)(y,w)=\varepsilon^{ik}\phi_1(e_i,e_k)\langle y,w\rangle-\varepsilon^{ik}\phi_1(y,e_k)\langle e_i,w\rangle\\&\quad-\varepsilon^{ik}\phi_1(e_i,Je_k)\langle Jy,w\rangle+\varepsilon^{ik}\phi_1(y,Je_k)\langle Je_i,w\rangle-2\varepsilon^{ik}\phi_1(e_i,Jy)\langle Je_k,w\rangle\\&\quad=0-\phi_1(y,w)-0-\phi_1(y,JJw)+2\phi_1(Jw,Jy)=2\phi_1(y,w),\\&(\rho_{13}\sigma_3\phi_3)(y,w)=\varepsilon^{ik}\phi_3(e_i,e_k)\langle y,w\rangle-\varepsilon^{ik}\phi_3(y,e_k)\langle e_i,w\rangle\\&\quad+2\varepsilon^{ik}\phi_3(e_i,y)\langle e_k,w\rangle-\varepsilon^{ik}\phi_3(e_i,Je_k)\langle Jy,w\rangle+\varepsilon^{ik}\phi_3(y,Je_k)\langle Je_i,w\rangle\\&\quad=0-\phi_3(y,w)+2\phi_3(w,y)-0-\phi_3(y,JJw)=-2\phi_3(y,w).\qquad\square\end{aligned}$$

We use Theorem 1.5 to give a $\mathcal{U}^\star$ module decomposition into irreducible and inequivalent $\mathcal{U}^\star$ modules (where as always we delete W_{11} if $m=4$):

$$\mathcal{K}_+\cap\ker(\rho)=S_{0,+^2}\oplus\Lambda^2_{0,+}\oplus W_9\oplus W_{10}\oplus W_{11}\,.$$

Let W_7 (resp. W_8) be the submodule of $\mathcal{K}_+\cap\ker(\rho)$ which is isomorphic as a $\mathcal{U}^\star$ module to $S^2_{0,+}$ (resp. $\Lambda^2_{0,+}$) under the map of ρ_{13}. Let π_7 (resp. π_8) be orthogonal projection on W_7 (resp. on W_8). Let $\rho_{13,a}$ (resp. $\rho_{13,s}$) be the alternating (resp. symmetric) part of ρ_{13}.

Lemma 3.2.

(1) $\pi_7=-\frac{1}{m(m+4)}\{2\sigma_1+(m+2)\vartheta\}\rho_{13,s}$.

(2) $\pi_8=-\frac{1}{m(m+4)}\{-2\sigma_3+(m+2)\vartheta\}\rho_{13,a}$.

Proof. We show that π_7 and π_8 split the action of ρ_{13} on $\mathcal{K}_+\cap\ker(\rho)$ by using Lemma 3.1 to see:

$$\begin{aligned}&\rho\pi_7\phi_1=-\tfrac{1}{m^2+4m}\{-(m+2)2+2(m+2)\}\phi_1=0,\\&\rho_{13}\pi_7\phi_1=-\tfrac{1}{m^2+4m}\{4-(m+2)^2\}\phi_1=\phi_1,\end{aligned}$$

$$\rho\pi_8\phi_3 = -\tfrac{1}{m^2+4m}\{(m+2)2-2(m+2)\}\phi_3 = 0,$$
$$\rho_{13}\pi_8\phi_3 = -\tfrac{1}{m^2+4m}\{4-(m+2)^2\}\phi_3 = \phi_3. \qquad \square$$

4. The conjugate tensor

Define the conjugate tensor A^* by setting:

$$A^*(x,y,z,w) := A(x,y,z,Jw)\,.$$

Lemma 4.1. *The map $T : A \to A^*$ satisfies:*

(1) $T^2 = -\operatorname{id}$.
(2) *T is a $\mathrm{GL}_{\mathbb{C}}^\star$ module morphism intertwining the module $\mathcal{K}_+ \cap \ker(\rho)$ with the module $\{\mathcal{K}_+ \cap \ker(\rho)\} \otimes \chi$.*
(3) *T is a $\mathcal{U}^\star$ module morphism which intertwines W_9 with $W_{10}\otimes\chi$, which intertwines W_7 with $W_8\chi$, and which intertwines W_{11} with $W_{11}\otimes\chi$.*

Remark 4.2. Since J appears an odd number of times in the definition of T, it is necessary to introduce the $\mathbb{Z}_2$ valued representation χ to take this into account. Since χ^2 is the trivial representation, this result also yields that T intertwines W_{10} with $W_9 \otimes \chi$ and that T intertwines W_8 with $W_7 \otimes \chi$.

Proof. Assertion (1) is immediate. Let $A \in \mathcal{K}_+ \cap \ker(\rho)$. By expressing

$$\begin{aligned} A^*(x,y,z,w) &= A(x,y,z,Jw) = A(x,y,Jz,JJw) \\ &= -A(x,y,Jz,w), \end{aligned}$$

we see that $\rho(A^*)(y,z) = -\rho(A)(y,Jz)$ and thus T preserves $\ker(\rho)$. It is immediate that A^* satisfies the Bianchi identity and that

$$A^*(x,y,Jz,Jw) = A(x,y,Jz,JJw) = A(x,y,z,Jw) = A^*(x,y,z,w),$$
$$A^*(Jx,Jy,z,w) = A(Jx,Jy,z,Jw) = A(x,y,z,Jw) = A^*(x,y,z,w).$$

Assertion (2) now follows. If $\psi \in \otimes^2 V^*$, we define $T\psi(x,y) := \psi(x,Jy)$. It is then immediate that $\rho_{13}TA = T\rho_{13}A$. Consequently T preserves the subspace $\ker(\rho_{13}) \cap \mathcal{K}_+ = W_9 \oplus W_{10} \oplus W_{11}$. We have:

$$\phi \in S^2_{0,+} \quad \Rightarrow \quad T\phi \in \Lambda^2_{0,+} \quad \text{and} \quad \psi \in \Lambda^2_{0,+} \quad \Rightarrow \quad T\psi \in S^2_{0,+}.$$

We see that T intertwines the representation W_9 with $W_{10}\otimes\chi$ by applying these relations to the last indices of a 4-tensor. Since T is an isometry, T intertwines W_{11} with $W_{11}\otimes\chi$ since W_{11} is the orthogonal complement of $W_9 \oplus W_{10}$ in the module $\mathcal{K}_+ \cap \ker(\rho) \cap \ker(\rho_{13})$. Since $W_7 \oplus W_8$ is the orthogonal complement of $W_9 \oplus W_{10} \oplus W_{11}$ in $\mathcal{K}_+ \cap \ker(\rho)$, T preserves the subspace $W_7 \oplus W_8$. Since T interchanges $S^2_{0,+}$ and $\Lambda^2_{0,+}$ and since T commutes with ρ_{13}, T interchanges the subspaces W_7 and W_8 and consequently intertwines the representation W_7 with $W_8 \otimes \chi$. $\square$

Lemma 4.3. *Let π_i for $i = 9, 10, 11$ be orthogonal projection on the $\mathcal{U}^\star$ modules W_i. Let $A \in \mathcal{K}_+ \cap \ker(\rho)$.*

(1) *If $\rho_{13}(A) \in \Lambda^2_{0,+}$, then $\pi_9(A)(x,y,z,w)$*
$$= \tfrac14\{A(x,y,z,w) + A(y,x,w,z) + A(z,w,x,y) + A(w,z,y,x)\}.$$
(2) *If $\rho_{13}(A) \in S^2_{0,+}$, then $\pi_{10}(A)(x,y,z,w) = -\frac14\{A(x,y,z,JJw)$*
$$+A(y,x,Jw,Jz) + A(z,Jw,x,Jy) + A(Jw,z,y,Jx)\}.$$
(3) $\pi_{11}(A) = \operatorname{id} - \pi_9 - \pi_{10}$.

Proof. Clearly $\pi_9(A)$ is anti-symmetric in (x,y). We verify that $\pi_9(A)$ satisfies the Bianchi identity and show $\pi_9(A) \in \mathfrak{A}$ by computing:

$$\begin{aligned}
&\pi_9(A)(x,y,z,w)+\pi_9(A)(y,z,x,w)+\pi_9(A)(z,x,y,w)\\
&\quad=\tfrac14\{A(x,y,z,w)+A(y,x,w,z)+A(z,w,x,y)+A(w,z,y,x)\}\\
&\quad+\tfrac14\{A(y,z,x,w)+A(z,y,w,x)+A(x,w,y,z)+A(w,x,z,y)\}\\
&\quad+\tfrac14\{A(z,x,y,w)+A(x,z,w,y)+A(y,w,z,x)+A(w,y,x,z)\}\\
&\quad=\tfrac14\{A(w,z,y,x)+A(z,y,w,x)+A(y,w,z,x)\}\\
&\quad+\tfrac14\{A(z,w,x,y)+A(w,x,z,y)+A(x,z,w,y)\}\\
&\quad+\tfrac14\{A(y,x,w,z)+A(x,w,y,z)+A(w,y,x,z)\}\\
&\quad+\tfrac14\{A(x,y,z,w)+A(y,z,x,w)+A(z,x,y,w)\}=0.
\end{aligned}$$

We show $\pi_9(A)\in\mathcal{K}_+$ by comparing:

$$\begin{aligned}
&\pi_9(A)(x,y,z,w)\\
&\quad=\tfrac14\{A(x,y,z,w)+A(y,x,w,z)+A(z,w,x,y)+A(w,z,y,x)\},\\
&\pi_9(A)(x,y,Jz,Jw)\\
&\quad=\tfrac14\{A(x,y,Jz,Jw)+A(y,x,Jw,Jz)+A(Jz,Jw,x,y)+A(Jw,Jz,y,x)\},\\
&\pi_9(A)(Jx,Jy,z,w)\\
&\quad=\tfrac14\{A(Jx,Jy,z,w)+A(Jy,Jx,w,z)+A(z,w,Jx,Jy)+A(w,z,Jy,Jx)\}.
\end{aligned}$$

As $\pi_9(A)$ is anti-symmetric in the last two indices, $\rho(\pi_9(A))=-\rho_{13}(\pi_9(A))$. We assume that $\rho(A)=0$ and that $\rho_{13}(A)$ is anti-symmetric. We show $\pi_9(A)\in\ker(\rho)$ and therefore that $\pi_9(A)$ takes values in W_9 by computing:

$$\begin{aligned}
&\rho(\pi_9(A))(y,z)=\tfrac14\varepsilon^{il}A(e_i,y,z,e_l)+\tfrac14\varepsilon^{il}A(y,e_i,e_l,z)\\
&\quad+\tfrac14\varepsilon^{il}A(z,e_l,e_i,y)+\tfrac14\varepsilon^{il}A(e_l,z,y,e_i)\\
&\quad=\tfrac14\{\rho(y,z)-\rho_{13}(y,z)-\rho_{13}(z,y)+\rho(z,y)\}=0.
\end{aligned}$$

Suppose A is anti-symmetric in (z,w). Then it is easily checked that $A\in\mathfrak{R}$ and hence $\pi_9(A)(x,y,z,w)=A(x,y,z,w)$. This completes the proof of Assertion (1).

By Lemma 4.1, T maps the subspace W_9 to the subspace W_{10}; the factor of χ is only added to take into account the equivariance and plays no role in the analysis. Since $T^{-1}=-T$ and since T is an isometry, we have therefore that $-T\pi_9T=\pi_{10}$; Assertion (2) now follows from Assertion (1); $T\rho_{13}=\rho_{13}T$ and T interchanges the subspaces $\Lambda^2_{0,+}$ with $S^2_{0,+}$. Assertion (3) is immediate from Assertions (1) and (2) and from Theorem 1.5. □

5. The proof of Theorem 1.2

As noted in Remark 2.3, we may complete the proof of Theorem 1.2, by showing:

Lemma 5.1. $\ker(\rho)\cap\mathcal{K}_+$ *is an irreducible* $\mathrm{GL}^\star_{\mathbb{C}}$ *module.*

Proof. We suppose to the contrary that ξ is a non-trivial proper $\mathrm{GL}^\star_{\mathbb{C}}$ submodule of $\ker(\rho)\cap\mathcal{K}_-$. We introduce an auxiliary Hermitian inner product $\langle\cdot,\cdot\rangle$. We apply Theorem 1.5. The modules $\{W_7,W_8,W_9,W_{10},W_{11}\}$ are inequivalent and irreducible $\mathcal{U}^\star$ modules (we delete W_{11} from consideration if $m=4$). Thus there is a set of indices $I\subset\{7,8,9,10,11\}$ so:

$$\xi=\oplus_{i\in I}W_i\,.$$

We choose an orthonormal basis $\{e_1,f_1,...,e_{\bar m},f_{\bar m}\}$ for V so $Je_i=f_i$ and $Jf_i=-e_i$. All 4-tensors considered in the proof of Lemma 5.1 will be anti-symmetric in the first 2 indices.

5.1. **Suppose that $W_9 \subset \xi$.** . Let A be determined by the relations:

$$\begin{aligned}
&A(e_1,f_1,e_1,f_2)=-1,\ A(e_1,f_1,f_1,e_2)=\ \ 1,\\
&A(e_1,f_1,e_2,f_1)=-1,\ A(e_1,f_1,f_2,e_1)=\ \ 1,\\
&A(e_1,f_2,e_1,f_1)=-1,\ A(e_1,f_2,f_1,e_1)=\ \ 1,\\
&A(e_1,f_2,e_2,f_2)=\ \ 1,\ A(e_1,f_2,f_2,e_2)=-1,\\
&A(f_1,e_2,e_1,f_1)=\ \ 1,\ A(f_1,e_2,f_1,e_1)=-1,\\
&A(f_1,e_2,e_2,f_2)=-1,\ A(f_1,e_2,f_2,e_2)=\ \ 1,\\
&A(e_2,f_2,e_1,f_2)=\ \ 1,\ A(e_2,f_2,f_1,e_2)=-1,\\
&A(e_2,f_2,e_2,f_1)=\ \ 1,\ A(e_2,f_2,f_2,e_1)=-1.
\end{aligned}$$

It is then immediate by inspection that $A\in W_9$. Let

$$\begin{array}{ll}
g_{1,\varepsilon}(e_i):=\left\{\begin{array}{lll}\varepsilon e_1&\text{if}&i=1\\ e_i&\text{if}&i\neq1\end{array}\right\}, & g_{1,\varepsilon}(e^i):=\left\{\begin{array}{lll}\varepsilon^{-1}e^1&\text{if}&i=1\\ e^i&\text{if}&i\neq1\end{array}\right\},\\
g_{1,\varepsilon}(f_i):=\left\{\begin{array}{lll}\varepsilon f_1&\text{if}&i=1\\ f_i&\text{if}&i\neq1\end{array}\right\}, & g_{1,\varepsilon}(f^i):=\left\{\begin{array}{lll}\varepsilon^{-1}f^1&\text{if}&i=1\\ f^i&\text{if}&i\neq1\end{array}\right\}.
\end{array}$$

Since ξ is a finite dimensional linear subspace, it is closed. Consequently

$$B_1:=\lim_{\varepsilon\to0}\varepsilon g_{1,\varepsilon}^*A\in\xi\,.$$

The non-zero components of B_1 and ρ_{13} are determined by:

$$\begin{array}{rr}
B_1(e_2,f_2,e_2,f_1)=1, & B_1(e_2,f_2,f_2,e_1)=-1,\\
\rho_{13}(B_1)(e_2,e_1)=1, & \rho_{13}(B_1)(f_2,f_1)=1\,.
\end{array}$$

By interchanging the roles of $\{e_1,f_1\}$ and $\{e_2,f_2\}$ we can create an element $B_2\in\xi$ with

$$\begin{array}{rr}
B_2(e_1,f_1,e_1,f_2)=1, & B_2(e_1,f_1,f_1,e_2)=-1,\\
\rho_{13}(B_2)(e_1,e_2)=1, & \rho_{13}(B_2)(f_1,f_2)=1.
\end{array}$$

Thus B_1+B_2 has a non-zero component in W_7 and B_1-B_2 has a non-zero component in W_8. This shows that:

$$W_9\subset\xi\quad\Rightarrow\quad W_7\oplus W_8\subset\xi\,.$$

Let $B_i^*:=TB_i$. We study $\pi_{10}(B_1+B_2)$ by examining $\pi_9(B_1^*+B_2^*)$:

$$\begin{array}{rr}
(B_1^*+B_2^*)(e_1,f_1,e_1,e_2)=1, & (B_1^*+B_2^*)(e_1,f_1,f_1,f_2)=1,\\
(B_1^*+B_2^*)(e_2,f_2,e_2,e_1)=1, & (B_1^*+B_2^*)(e_2,f_2,f_2,f_1)=1,\\
\rho_{13}(B_1^*+B_2^*)(f_1,e_2)=1, & \rho_{13}(B_1^*+B_2^*)(e_2,f_1)=-1,\\
\rho_{13}(B_1^*+B_2^*)(f_2,e_1)=1, & \rho_{13}(B_1^*+B_2^*)(e_1,f_2)=-1.
\end{array}$$

Since $\rho_{13}(B_1^*+B_2^*)$ is anti-symmetric, we have by Lemma 4.3 that:

$$\pi_9(B_1^*+B_2^*)(e_1,f_1,e_1,e_2)=\tfrac14\,.$$

Consequently $\pi_{10}(B_1+B_2)\neq0$. This implies:

$$W_9\subset\xi\quad\Rightarrow\quad W_{10}\subset\xi\,.$$

Suppose $m\geq6$. Set

$$\begin{array}{ll}
g_{2,\varepsilon}(e_i):=\left\{\begin{array}{lll}e_3-\varepsilon e_1&\text{if}&i=3\\ e_i&\text{if}&i\neq3\end{array}\right\}, & g_{2,\varepsilon}(e^i):=\left\{\begin{array}{lll}e^1+\varepsilon e^3&\text{if}&i=1\\ e^i&\text{if}&i\neq1\end{array}\right\},\\
g_{2,\varepsilon}(f_i):=\left\{\begin{array}{lll}f_3-\varepsilon f_1&\text{if}&i=1\\ f_i&\text{if}&i\neq3\end{array}\right\}, & g_{2,\varepsilon}(f^i):=\left\{\begin{array}{lll}f^1+\varepsilon f^3&\text{if}&i=1\\ f^i&\text{if}&i\neq1\end{array}\right\}.
\end{array}$$

Let $B_3:=\partial_\varepsilon\{g_{2,\varepsilon}^*A\}|_{\varepsilon=0}$. We then have:

$$\begin{aligned}
&B_3(e_1,f_1,e_2,f_3)=-1,\ B_3(e_1,f_1,f_2,e_3)=1,\\
&B_3(e_1,f_2,e_1,f_3)=-1,\ B_3(e_1,f_2,f_1,e_3)=1,\\
&B_3(f_1,e_2,e_1,f_3)=1,\ B_3(f_1,e_2,f_1,e_3)=-1,\\
&B_3(e_2,f_2,e_2,f_3)=1,\ B_3(e_2,f_2,f_2,e_3)=-1.
\end{aligned}$$

We use Lemma 4.3 to see $|\pi_9 B_3(e_1,f_1,e_2,e_3)|\le\frac14$ and $|\pi_{10}(e_1,f_1,e_2,e_3)|\le\frac14$. Since $B_3\in\ker(\rho_{13})$, we have $|\pi_{11}B_3(e_1,f_1,e_2,e_3)|\ge\frac12$ and thus $W_{11}\subset\xi$. We summarize our conclusions:

$$W_9\subset\xi\quad\Rightarrow\quad\xi=\mathcal{K}_+\cap\ker(\rho)\,.$$

5.2. **Suppose that $W_7\subset\xi$.** . We clear the previous notation. Let

$$\phi:=e^1\otimes e^2+e^2\otimes e^1+f^1\otimes f^2+f^2\otimes f^1\in S^2_{0,+}\,.$$

We use Lemma 3.2 to find $A\in W_7$ so that $\rho_{13}A=\phi$. We shall not compute all the terms in A as this would be a bit of a bother and shall content ourselves with determining just a few terms. We compute:

$$\begin{aligned}
&\langle\sigma_1\phi(e_2,f_2)e_2,e_1\rangle=0,\qquad\langle\sigma_1\phi(e_2,f_2)e_2,f_1\rangle=0,\\
&\vartheta(\phi)(e_2,f_2,e_2,e_1):=\phi(e_2,e_1)\langle f_2,e_2\rangle-\phi(f_2,e_1)\langle e_2,e_2\rangle\\
&\quad+\phi(e_2,Je_1)\langle f_2,Je_2\rangle-\phi(f_2,Je_1)\langle e_2,Je_2\rangle-2\phi(e_2,Je_1)\langle e_2,Jf_2\rangle=0,\\
&\vartheta(\phi)(e_2,f_2,e_2,f_1):=\phi(e_2,f_1)\langle f_2,e_2\rangle-\phi(f_2,f_1)\langle e_2,e_2\rangle\\
&\quad+\phi(e_2,Jf_1)\langle f_2,Je_2\rangle-\phi(f_2,Jf_1)\langle e_2,Je_2\rangle\\
&\quad-2\phi(e_2,Jf_1)\langle e_2,Jf_2\rangle=0-1-1-0-2\ne0.
\end{aligned}$$

$$0=c_1:=A(e_2,f_2,e_2,e_1),\qquad 0\ne c_2:=A(e_2,f_2,e_2,f_1).$$

Let $\Phi\in\mathcal{U}$ be defined by:

$$\Phi e_i:=\left\{\begin{array}{rll}-e_1&\text{if}&i=1\\ e_i&\text{if}&i>1\end{array}\right\},\qquad\Phi f_i:=\left\{\begin{array}{rll}-f_1&\text{if}&i=1\\ f_i&\text{if}&i>1\end{array}\right\}.$$

Since $\Phi^*\phi=-\phi$, we have $\Phi^*A=-A$. Thus the number of times that x_i is e_1 or f_1 is odd; similarly, the number of times that x_i is f_1 or f_2 is odd as well. Define $g_{\varepsilon_1,\varepsilon_2}\in\mathrm{GL}^\star_{\mathbb{C}}$ by setting:

$$g_{\varepsilon_1,\varepsilon_2}e_i=\left\{\begin{array}{lll}\varepsilon_1e_1&\text{if}&i=1\\ \varepsilon_2e_2&\text{if}&i=2\\ e_i&\text{if}&i\ge3\end{array}\right\},\quad g_{\varepsilon_1,\varepsilon_2}e^i=\left\{\begin{array}{lll}\varepsilon_1^{-1}e^1&\text{if}&i=1\\ \varepsilon_2^{-1}e^2&\text{if}&i=2\\ e^i&\text{if}&i\ge3\end{array}\right\},$$
$$g_{\varepsilon_1,\varepsilon_2}f_i=\left\{\begin{array}{lll}\varepsilon_1f_1&\text{if}&i=1\\ \varepsilon_2f_2&\text{if}&i=2\\ f_i&\text{if}&i\ge3\end{array}\right\},\quad g_{\varepsilon_1,\varepsilon_2}f^i=\left\{\begin{array}{lll}\varepsilon_1^{-1}f^1&\text{if}&i=1\\ \varepsilon_2^{-1}f^2&\text{if}&i=2\\ f^i&\text{if}&i\ge3\end{array}\right\}.$$

Expand $g^*_{\varepsilon_1,\varepsilon_2}A$ as a finite Laurent polynomial in $\{\varepsilon_1,\varepsilon_2\}$. As $g^*_{\varepsilon_1,\varepsilon_2}A\in\xi$, all the coefficient curvature tensors also belong to ξ. Let $B\in\xi$ be the coefficient of $\varepsilon_1^{-1}\varepsilon_2^3$ in $g^*_{\varepsilon_1,\varepsilon_2}A$;

$$B=\left\{\tfrac16\varepsilon_1\partial^3_{\varepsilon_2}g^*_{\varepsilon_1,\varepsilon_2}A\right\}\Big|_{\varepsilon_1=0,\varepsilon_2=0}\,.$$

The only (possibly) non-zero components of B are given by:

$$\begin{aligned}
&B(e_2,f_2,e_2,e_1)=A(e_2,f_2,e_2,e_1)=0,\\
&B(e_2,f_2,e_2,f_1)=A(e_2,f_2,e_2,f_1)=c_2,\\
&B(e_2,f_2,f_2,e_1)=-B(e_2,f_2,e_2,f_1)=-c_2,\\
&B(e_2,f_2,f_2,f_1)=B(e_2,f_2,e_2,e_1)=0.
\end{aligned}$$

We examine:

$$\rho_{13}(B)(e_2,e_1)=c_2 \quad \text{and} \quad \rho_{13}(B)(f_2,f_1)=c_2\,.$$

Interchanging the roles of the indices "1" and "2" is an isometry which preserves ϕ; this creates a tensor $\tilde{B}\in\xi$ so that

$$\begin{array}{ll}\tilde{B}(e_1,f_1,f_1,e_2)=-c_2, & \tilde{B}(e_1,f_1,e_1,f_2)=c_2,\\ \rho_{13}(\tilde{B})(e_1,e_2)=c_2, & \rho_{13}(\tilde{B})(f_1,f_2)=c_2.\end{array}$$

In particular $B-\tilde{B}$ has an anti-symmetric Ricci tensor so we may use Lemma 4.3 to compute

$$\pi_9(B-\tilde{B})(e_2,f_2,e_2,f_1)=\tfrac{1}{4}c_2\neq 0\,.$$

This implies $W_9\subset\xi$ and hence by Section 5.1,

$$W_7\subset\xi \quad\Rightarrow\quad W_9\subset\xi \quad\Rightarrow\quad \xi=\mathcal{K}_+\cap\ker(\rho)\,.$$

5.3. **Suppose that $m\geq 6$ and that $W_{11}\subset\xi$.** . Clear the previous notation. Set:

$$\begin{array}{ll}
A(e_1,e_2,e_1,e_3)=1, & A(e_1,e_2,f_1,f_3)=1,\\
A(e_1,f_2,e_1,f_3)=-1, & A(e_1,f_2,f_1,e_3)=1,\\
A(e_1,e_3,e_1,e_2)=-1, & A(e_1,e_3,f_1,f_2)=-1,\\
A(e_1,f_3,e_1,f_2)=1, & A(e_1,f_3,f_1,e_2)=-1,\\
A(f_1,e_2,e_1,f_3)=1, & A(f_1,e_2,f_1,e_3)=-1,\\
A(f_1,f_2,e_1,e_3)=1, & A(f_1,f_2,f_1,f_3)=1,\\
A(f_1,e_3,e_1,f_2)=-1, & A(f_1,e_3,f_1,e_2)=1,\\
A(f_1,f_3,e_1,e_2)=-1, & A(f_1,f_3,f_1,f_2)=-1\,.
\end{array}$$

We verify by inspection that $A\in\mathcal{K}_+\cap\ker(\rho)\cap\ker(\rho_{13})$. We study:

$$\begin{array}{rcl}\pi_9(A)(x,y,z,w) &=& \tfrac{1}{4}\{A(x,y,z,w)+A(y,x,w,z)\\ && \quad +A(z,w,x,y)+A(w,z,y,x)\}\,.\end{array}$$

Let U_1 denote the set of elements $\{e_2,f_2,e_3,f_3\}$. For $\pi_9(A)$ to be non-zero, either $x\in U_1$ or $y\in U_1$ and either $z\in U_1$ or $w\in U_1$. If x and z belong to U_1, then we have that $A(x,y,z,w)=-A(z,w,x,y)$ and that $A(y,x,w,z)=A(w,z,y,x)=0$. Thus $\pi_9A(x,y,z,w)=0$ in this special case. Since π_9A is anti-symmetric in the first 2 indices and in the last 2 indices, we see that $\pi_9A=0$ in the remaining cases. To examine π_{10}, we consider the dual tensor $A^*=TA$:

$$\begin{array}{ll}
A^*(e_1,e_2,e_1,f_3)=-1, & A^*(e_1,e_2,f_1,e_3)=1,\\
A^*(e_1,f_2,e_1,e_3)=-1, & A^*(e_1,f_2,f_1,f_3)=-1,\\
A^*(e_1,e_3,e_1,f_2)=1, & A^*(e_1,e_3,f_1,e_2)=-1,\\
A^*(e_1,f_3,e_1,e_2)=1, & A^*(e_1,f_3,f_1,f_2)=1,\\
A^*(f_1,e_2,e_1,e_3)=1, & A^*(f_1,e_2,f_1,f_3)=1,\\
A^*(f_1,f_2,e_1,f_3)=-1, & A^*(f_1,f_2,f_1,e_3)=1,\\
A^*(f_1,e_3,e_1,e_2)=-1, & A^*(f_1,e_3,f_1,f_2)=-1,\\
A^*(f_1,f_3,e_1,f_2)=1, & A^*(f_1,f_3,f_1,e_2)=-1\,.
\end{array}$$

Once again $x\in U_1$ and $z\in U_1$ implies $A^*(x,y,z,w)+A^*(z,w,x,y)=0$ while $A^*(y,x,w,z)=A^*(w,z,y,x)=0$. The argument given above to show that $\pi_9A=0$ then shows $\pi_9A^*=0$ and hence $\pi_{10}A=0$. Consequently since $\rho(A)=\rho_{13}(A)=0$, we may conclude that $A\in W_{11}$. Set:

$$\begin{array}{ll}
g_\varepsilon(e_i):=\left\{\begin{array}{lll}\varepsilon e_3 & \text{if} & i=3\\ e_i & \text{if} & i\neq 3\end{array}\right\}, & g_\varepsilon(e^i):=\left\{\begin{array}{lll}\varepsilon^{-1}e^3 & \text{if} & i=3\\ e^i & \text{if} & i\neq 3\end{array}\right\},\\
g_\varepsilon(f_i):=\left\{\begin{array}{lll}\varepsilon f_3 & \text{if} & i=3\\ f_i & \text{if} & i\neq 3\end{array}\right\}, & g_\varepsilon(f^i):=\left\{\begin{array}{lll}\varepsilon^{-1}f^3 & \text{if} & i=3\\ f^i & \text{if} & i\neq 3\end{array}\right\}.
\end{array}$$

We set $B := \lim_{\varepsilon\to 0} \varepsilon g_\varepsilon^* A \in \xi$. We see that the non-zero components of B are determined by:

$$\begin{aligned} B(e_1,e_2,e_1,e_3) &= 1,\ B(e_1,e_2,f_1,f_3) = 1,\\ B(e_1,f_2,e_1,f_3) &= -1,\ B(e_1,f_2,f_1,e_3) = 1,\\ B(f_1,e_2,e_1,f_3) &= 1,\ B(f_1,e_2,f_1,e_3) = -1,\\ B(f_1,f_2,e_1,e_3) &= 1,\ B(f_1,f_2,f_1,f_3) = 1. \end{aligned}$$

We verify that $\rho(B) = \rho_{13}(B) = 0$. We use Lemma 4.3 to see:

$$\begin{aligned} &\pi_9(B)(e_1,e_2,e_1,e_3) = \tfrac14 B(e_1,e_2,e_1,e_3) = \tfrac14,\\ &\pi_{10}(B)(e_1,e_2,e_1,e_3) = -\tfrac14 \pi_9(B^*)(e_1,e_2,e_1,f_3)\\ &\qquad = \tfrac14 B(e_1,e_2,e_1,e_3) = \tfrac14\,. \end{aligned}$$

We use Section 5.1 to see that if $m \geq 6$, then

$$W_{11} \subset \xi \quad\Rightarrow\quad W_9 \subset \xi \quad\Rightarrow\quad \xi = \mathcal{K}_+ \cap \ker(\rho)\,.$$

5.4. **Suppose that $W_{10} \subset \xi$.** . We use Lemma 4.1 to interchange the roles of W_9 and W_{10} and then apply the results of Section 5.1 to see:

$$W_{10} \subset \xi \quad\Rightarrow\quad W_9 \subset T\xi \quad\Rightarrow\quad T\xi = \mathcal{K}_+ \cap \ker(\rho) \quad\Rightarrow\quad \xi = \mathcal{K}_+ \cap \ker(\rho)\,.$$

5.5. **Suppose that $W_8 \subset \xi$.** . We use the duality operator and Section 5.2 to see:

$$W_8 \subset \xi \quad\Rightarrow\quad W_7 \subset T\xi \quad\Rightarrow\quad T\xi = \mathcal{K}_+ \cap \ker(\rho) \quad\Rightarrow\quad \xi = \mathcal{K}_+ \cap \ker(\rho)\,.$$

This completes the proof of Lemma 5.1 and thereby of all the assertions in this paper. □

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